two values of p, x', y', for each of the two values of e^2, and two values of λ and μ for each pair of values of p. We rejected the negative value of μ as leading to the same geometric results as its positive value. We thus had left eight solutions of the system of equations. The negative value of e may also be rejected as not giving essentially different geometric results. There will be left four solutions. Since the process of elimination consisted only in the addition, subtraction, and multiplication of the equations (division of equations not being employed), it follows that no set of finite solutions could have escaped. The remaining 2032 sets of solutions must therefore be partly or wholly infinite whatever may be the values of a, b, c, f, g, h. That the set of equations (17) may have a set of values of e, λ, μ, p, x', y' partly infinite appears as follows. Suppose a, b, c, f, g, h to have any finite values, then if λ, μ, p be supposed finite, and e, x', y' be supposed infinities of the first order the equations may be satisfied. Furthermore, if e, λ, μ, p be supposed infinites of the first order, and x', y' infinites of the second order, the equations may be satisfied.

THE HOMOGENEOUS EQUATION OF FOUR TERMS TO EXPRESS THE HOMOGRAPHIC DIVISION OF TWO STRAIGHT LINES.

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Chasles in his *Traité de Géométrie Supérieure* Art. 137, (Ed. 1852),* says: If two straight lines are divided homographically, and we take upon the one line two fixed points a, c, and upon the other, two fixed points b', d', homographic division will be expressed by the homogeneous equation

$$\frac{am}{cm}\cdot\frac{b'm'}{d'm'}+\lambda\frac{am}{cm}+\mu\frac{b'm'}{d'm'}+\nu=0.$$

His first demonstration (Art. 137) is indirect, and would not readily suggest itself, and the second demonstration (Art. 140) is cumbrous. The proposition may also be demonstrated in the following manner:—

By Art. 133, homographic division is expressed by the following equation:—

$$am\,.\,b'm'-am\,.\,b'J'-b'm'\,.\,aI+aI\,.\,b'a'=0,$$

in which I is the point on the first line corresponding to infinity on the second and J' the point on the second corresponding to infinity on the first. Dividing

*Art. 137, Ed. 1852 is 144, Ed. 1880; 133 is 139; 140 has been omitted.

by $aI \,.\, b'J'$, we have:—

$$\frac{am}{aI}\cdot\frac{b'm'}{b'J'}-\frac{am}{aI}-\frac{b'm'}{b'J'}+\frac{b'a'}{b'J'}=0.$$

Introducing the point at infinity on each line,

$$\left(\frac{am}{aI}\div\frac{\infty m}{\infty I}\right)\left(\frac{b'm'}{b'J'}\div\frac{\infty m'}{\infty J'}\right)-\frac{am}{aI}\div\frac{\infty m}{\infty I}-\frac{b'm'}{b'J'}\div\frac{\infty m'}{\infty J'}+\frac{b'a'}{b'J'}\div\frac{\infty a'}{\infty J'}=0.$$

This is an equation connecting a, m, I, ∞ of the first line and b', m', ∞, J' of the second. While the three points m, I, ∞ correspond to the three, m', ∞, J', the point a does not correspond to the point b'. We may, therefore, fix a and b' arbitrarily. The equation involves only anharmonic ratios; thus (Art. 20) it remains true when for the corresponding points, m, I, ∞; m', ∞, J', we substitute the corresponding points, m, d, c; m', d', c'. It is evident that we may fix c, c' and d, d' arbitrarily as corresponding points. Hence the preceeding equation becomes

$$\left(\frac{am}{ad}\div\frac{cm}{cd}\right)\left(\frac{b'm'}{b'c'}\div\frac{d'm'}{d'c'}\right)-\frac{am}{ad}\div\frac{cm}{cd}-\frac{b'm'}{b'c'}\div\frac{d'm'}{d'c'}+\frac{b'a'}{b'c'}\div\frac{d'a'}{d'c'}=0,$$

in which a, c are any two fixed points on the one line and b', d' any two fixed points on the other. We may write the equation thus:—

$$\frac{am}{cm}\cdot\frac{b'm'}{d'm'}\cdot\frac{cd}{ad}\cdot\frac{d'c'}{b'c'}-\frac{am}{cm}\cdot\frac{cd}{ad}-\frac{b'm'}{d'm'}\cdot\frac{d'c'}{b'c'}-\frac{b'a'}{d'a'}\cdot\frac{d'c'}{b'c'}=0.$$

Dividing this equation by $\frac{cd}{ad}\cdot\frac{d'c'}{b'c'}$, we have:—

$$\frac{am}{cm}\cdot\frac{b'm'}{d'm'}-\frac{am}{cm}\cdot\frac{b'c'}{d'c'}-\frac{b'm'}{d'm'}\cdot\frac{ad}{cd}+\frac{b'a'}{d'a'}\cdot\frac{ad}{cd}=0$$

or

$$\frac{am}{cm}\cdot\frac{b'm'}{d'm'}+\lambda\frac{am}{cm}+\mu\frac{b'm'}{d'm'}+\nu=0,$$

the required equation.

Comparing the last two equations, we have the values of λ, μ, ν deduced in Art. 138.